Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE MAGIC SQUARE.

I.

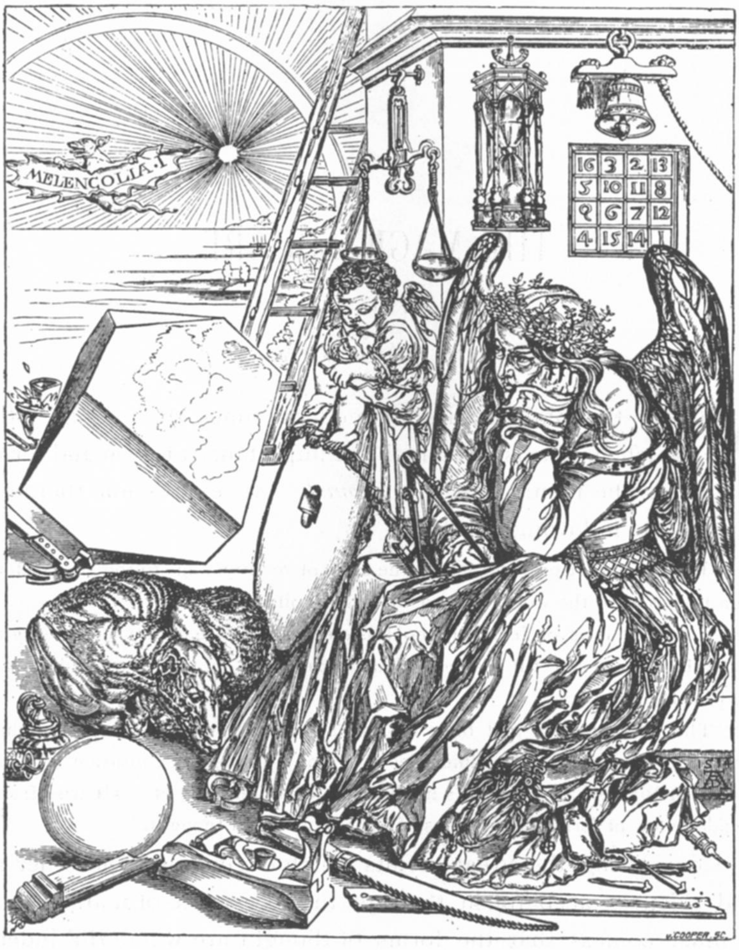
INTRODUCTORY.

AMONG the philosophies of modern times there is no other which emphasises so much the importance of form and formal thought as the monism of *The Monist.* An expression thereof is found in the following passages :

> "The order that prevails among the facts of reality is due to the laws of form. Upon the order of the world depends its cognisability.
>
> ". . . . The laws of form are no less eternal than are matter and energy and 'Verily I say unto you, till heaven and earth pass, one jot or one tittle shall in no wise pass from the law !'
>
> "The laws of form and their origin have been a puzzle to all philosophers. 'Ay, there's the rub !' The difficulties of Hume's problem of causation, of Kant's *a priori,* of Plato's ideas, of Mill's method of deduction, etc., etc., all arise from a one-sided view of form and the laws of form and formal thought."

Considering the great results which engineering and other applied sciences accomplish through the assistance of mathematics, we must confess that the forms of thought are wonderful indeed, and it is not at all astonishing that the primitive thinkers of mankind when the importance of the laws of formal thought in some way or another first dawned on their minds, attributed magic powers to numbers and geometrical figures.

We shall devote the following pages to a brief review of magic squares, the consideration of which has made many a man believe in mysticism. And yet there is no mysticism about them unless we either consider everything mystical, even that twice two is four, or join the sceptic in his exclamation that we can truly not



ALBERT DÜRER'S ENGRAVING

MELANCHOLY

OR THE

GENIUS OF THE INDUSTRIAL SCIENCE OF MECHANICS.

know whether twice two might not be five in other spheres of the universe.

The author of the short article on "Magic Squares" in the English Cyclopædia (Vol. III, p. 415), presumably Prof. De Morgan, says:

> "Though the question of magic squares be in itself of no use, yet it belongs to a class of problems which call into action a beneficial species of investigation. Without laying down any rules for their construction, we shall content ourselves with destroying their magic quality, and showing that the non-existence of such squares would be much more surprising than their existence."

This is the point. There obtains a symphonic harmony in mathematics which is the more startling the more obvious and self-evident it appears to him who understands the laws that produce this symphonic harmony.

* * *

On the wood-cut named "Melancholia"* of the famous Nuremberg painter, Albrecht Dürer, is found among a number of other emblems, which the reader will notice in our reproduction of the cut, the subjoined square. This arrangement of the sixteen natural numbers from 1 to 16 possesses the remarkable property that the same sum 34 will always be obtained whether we add together the four figures of any of the horizontal rows or the four of any vertical row or the four which lie in either of the two diagonals. Such an arrangement of numbers is termed a magic square, and the square which we have reproduced above is *the first magic square which is met with in the Christian Occident.*

---

* The term melancholy meant in Dürer's time, as it did also in Shakespeare's and Milton's, "thought or thoughtfulness." Says Milton in *Il Penseroso*:

"Hail, thou Goddess, sage and holy,
Hail divinest melancholy
Whose saintly visage is too bright
To hit the sense of human sight,
And therefore to our weaker view
O'erlaid with black, staid Wisdom's hue.—I, 12.

Thought that does not lead to action produces a gloomy state of mind. Thoughtfulness which cannot find a way out of itself is that melancholy which engenders weakness,—a truth which is illustrated in Hamlet. Shakespeare still uses the words thought and melancholy as synonyms, saying:

"The native hue of resolution
Is sicklied o'er with the pale cast of thought."

Dürer's melancholy does not represent the gloominess of thought, but the power of invention. Soberness and even a certain sadness are considered only as an element of this melancholy, but on the whole the genius of thought appears bright, self-possessed, and strong.

Dürer represents the Science of Mechanical Invention as a winged female figure musing over some problem. Scattered on the floor around her lie some of the simple tools used in the sixteenth century. A ladder leans against the house, that assists in climbing otherwise inaccessible heights. A scale, an hour-glass, a bell, and the magic square are hanging on the wall behind her.

At a distance a bat-like creature, being the gloom of melancholy, hovers in the air like a dark cloud, but the sun rises above the horizon, and at the happy middle between these two extremes stands the rainbow of serene hope and cheerful confidence.

1	14	15	4
12	7	6	9
8	11	10	5
13	2	3	16

Fig. 1.

Like chess and many of the problems founded on the figure of the chess-board, the problem of constructing a magic square also probably traces its origin to Indian soil. From there the problem found its way among the Arabs, and by them it was brought to the Roman Orient. Finally, since Albrecht Dürer's time, the scholars of Western Europe also have occupied themselves with methods for the construction of squares of this character.

The oldest and the simplest magic square consists of the quadratic arrangement of the nine numbers from 1 to 9 in such a manner that the sum of each horizontal, vertical, or diagonal row, always remains the same, namely 15. This square is the adjoined.

2	7	6
9	5	1
4	3	8

Fig. 2.

Here, we will find, 15 always comes out whether we add 2 and 7 and 6, or 9 and 5 and 1, or 4 and 3 and 8, or 2 and 9 and 4, or 7 and 5 and 3, or 6 and 1 and 8, or 2 and 5 and 8, or 6 and 5 and 4.

The question naturally presents itself, whether this condition of the constant equality of the added sum also remains fulfilled when the numbers are assigned different places. It may be easily shown however that 5 necessarily must occupy the middle place, and that the even numbers must stand in the corners. This being so, there are but 7 additional arrangements possible, which differ from the arrangement above given and from one another only in the respect that the rows at the top, at the left, at the bottom, and at the right, exchange places with one another and that in addition a mirror be imagined present with each arrangement. So too from Dürer's square of 4 times 4 places, by transpositions, a whole set of new correct squares may be formed. A magic square of the 4 times 4 numbers from 1 to 16 is formed in the simplest manner as follows. We inscribe the numbers from 1 to 16 in their natural order in the squares, thus:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 3.

We then leave the numbers in the four corner-squares, viz. 1, 4, 13, 16, as well also as the numbers in the four middle-squares, viz. 6, 7, 10, 11, in their original places; and in the place of the remaining eight numbers, we write the complements of the same with respect to 17: thus 15 instead of 2, 14 instead of 3, 12 instead of 5, 9 instead of 8, 8 instead of 9, 5 instead of 12, 3 instead of 14, and 2 instead of 15. We obtain thus the magic square

34				34
1	15	14	4	=34
12	6	7	9	=34
8	10	11	5	=34
13	3	2	16	=34
34	34	34	34	

Fig. 4.

from which the same sum 34 always results. It is an interesting property of this square that any four numbers which form a rectangle or square about the centre also always give the same sum 34; for example, 1, 4, 13, 16, or 6, 7, 10, 11, or 15, 14, 3, 2, or 12, 9, 5, 8, or 15, 8, 2, 9, or 14, 12, 3, 5. We may easily convince ourselves that this square is obtainable from the square of Dürer by interchanging with one another the two middle vertical rows.

II.

EARLY METHODS FOR THE CONSTRUCTION OF ODD-NUMBERED SQUARES.

Since early times rules have also been known for the construction of magic squares of more than 3 times 3, or 4 times 4 spaces. In the first place, it is easy to calculate the sum which in the case of any given number of cells must result from the addition of each row. We take the determinate number of cells in each side of the square which we have to fill, multiply that number by itself, add 1, again multiply the number thus obtained by the number of the cells in each side, and, finally, divide the product by 2. Thus, with 4 times 4 cells or squares, we get: 4 times 4 are 16, 16 and 1 are 17, and one half of 17 times 4 is 34. Similarly, with 5 times 5 squares, we get: 5 times 5 are 25, and 1 makes 26, and the half of 26 times 5 is 65. Analogously, for 6 times 6 squares the summation 111 is obtained, for 7 times 7 squares 175, for 8 times 8 squares 260, for 9 times 9 squares 369, for 10 times 10 squares 505, and so on. The Hindu rule for the construction of magic squares whose roots are odd, may be enunciated as follows: To start with, write 1 in the centre of the topmost row, then write 2 in the lowest space of the vertical column next adjacent to the right, and then so inscribe the remaining numbers in their natural order in the squares diagonally upwards towards the right, that on reaching the right-hand margin

175							175
30	39	48	1	10	19	28	=175
38	47	7	9	18	27	29	=175
46	6	8	17	26	35	37	=175
5	14	16	25	34	36	45	=175
13	15	24	33	42	44	4	=175
21	23	32	41	43	3	12	=175
22	31	40	49	2	11	20	=175
175	175	175	175	175	175	175	

Fig. 5.

the inscription shall be continued from the left-hand margin in the row just above, and on reaching the upper margin shall be continued from the lower margin in the column next adjacent to the right, noting that whenever we are arrested in our progress by a square already occupied we are to fill out the square next beneath the one we have last filled. In this manner, for example, the last preceding square of 7 times 7 cells is formed, in which the reader is requested to follow the numbers in their natural sequence (Fig. 5).

For the next further advancements of the theory of magic squares and of the methods for their construction we are indebted to the Byzantian Greek, Moschopulus, who lived in the fourteenth century; also, after Albrecht Dürer who lived about the year 1500, to the celebrated arithmetician Adam Riese, and to the mathematician Michael Stifel, which two last lived about 1550. In the seventeenth century Bachet de Méziriac, and Athanasius Kircher employed themselves on magic squares. About 1700, finally, the French mathematicians De la Hire and Sauveur made considerable contributions to the theory. In recent times mathematicians have concerned themselves much less about magic squares, as they have indeed about mathematical recreations generally. But quite recently the Brunswick mathematician Scheffler has put forth his own and other's studies on this subject in an elegant form.

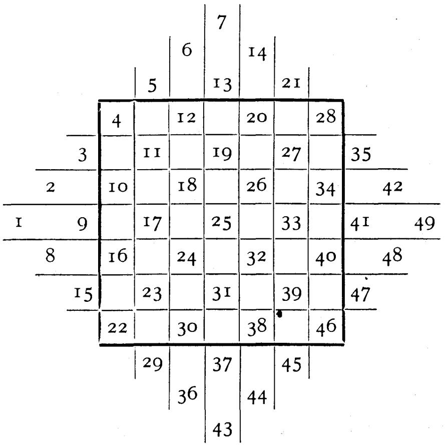


Fig. 6.

The best known of the various methods of constructing magic squares of an odd number of cells is the following. First write the numbers in diagonal succession as in the preceding diagram (Fig. 6). After 25 cells of the square of 49 cells which we have to fill out, have thus been occupied, transfer the six figures found outside each side of the square, without changing their configuration, into the empty cells of the side directly opposite. By this method, which we owe to Bachet de Méziriac, we obtain the following magic square of the numbers from 1 to 49:

4	29	12	37	20	45	28
35	11	36	19	44	27	3
10	42	18	43	26	2	34
41	17	49	25	1	33	9
16	48	24	7	32	8	40
47	23	6	31	14	39	15
22	5	30	13	38	21	46

Fig. 7.

III.

MODERN MODES OF CONSTRUCTION OF ODD-NUMBERED SQUARES.

The reader will justly ask whether there do not exist other correct magic squares which are constructed after a different method from that just given, and whether there do not exist modes of construction which will lead to all the imaginable and possible magic squares of a definite number of cells. A general mode of construction of this character was first given for odd-numbered squares by De la Hire, and recently perfected by Professor Scheffler.

To acquaint ourselves with this general method, let us select as our example a square of 5. First we form two auxiliary squares. In the first we write the numbers from 1 to 5 five times; and in the second, five times, the following multiples of five, viz.: 0, 5, 10, 15, 20. It is clear now that by adding each of the numbers of the series from 1 to 5 with each of the numbers 0, 5, 10, 15, 20, we shall get all the 25 numerals from 1 to 25. All that additionally remains to be done therefore, is, so to inscribe the numbers that by the addition of the two numbers in any two corresponding cells each combination shall come out once and only once ; . and further that in each horizontal, vertical, and diagonal row in each auxiliary square each number shall once appear. Then the required sum of 65 must necessarily result in every case, because the numbers from 1 to 5 added together make 15, and the numbers 0, 5, 10, 15, 20 make 50.

We effect the required method of inscription by imagining the numbers 1, 2, 3, 4, 5 (or 0, 5, 10, 15, 20) arranged in cyclical succession, that is 1 immediately following upon 5, and, starting from any number whatsoever, by skipping each time either none or one or two or three etc. figures. Cycles are thus obtained of the first, the second, the third etc. orders ; for example 3 4 5 1 2 is a cycle of the first order, 2 4 1 3 5 is a cycle of the second order, 1 5 4 3 2 is a cycle of the fourth order, etc. The only thing then to be looked out for in the two auxiliary squares is, that the same "cycle" order be horizontally preserved in all the rows, that the same also happens for the vertical rows, but that the cycle order in the horizontal and vertical rows is different. Finally we have only additionally to take care that to the same numbers of the one auxiliary square not like numbers but *different* numbers correspond in the other auxiliary square, that is lie in similarly situated cells. The following auxiliary squares are, for example, thus possible :

3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
1	2	3	4	5

Fig. 8.

and

0	10	20	5	15
5	15	0	10	20
10	20	5	15	0
15	0	10	20	5
20	5	15	0	10

Fig. 9.

Adding in pairs the numbers which occupy similarly situated cells, we obtain the following correct magic square :

3	14	25	6	17
10	16	2	13	24
12	23	9	20	1
19	5	11	22	8
21	7	18	4	15

Fig. 10.

It will be seen that we are able thus to construct a very large number of magic squares of 5 times 5 spaces by varying in every possible manner the numbers in the two auxiliary squares. Furthermore, the squares thus formed possess the additional peculiarity, that every 5 numbers which fill out two rows that are parallel to a diagonal and lie on different sides of the diagonal also give the constant sum of 65. For example: 3 and 7, 11, 20, 24; or 10, 14 and 18, 22, 1. Altogether then the sum 65 is produced out of 20 rows or pairs of rows. On this peculiarity is dependent the fact that if we imagine an unlimited number of such squares placed by the side of, above, or beneath an initial one, we shall be able to obtain as many quadratic cells as we choose, so arranged that the square composed of any 25 of these cells will form a correct magic square, as the following figure will show :

2	13	24	10	16	2	13	24	10	16	2
9	20	1	12	23	9	20	1	12	23	9
11	22	8	19	5	11	22	8	19	5	11
18	4	15	21	7	18	4	15	21	7	18
25	6	17	3	14	25	6	17	3	14	25
2	13	24	10	16	2	13	24	10	16	2
9	20	1	12	23	9	20	1	12	23	9
11	22	8	19	5	11	22	8	19	5	11
18	4	15	21	7	18	4	15	21	7	18
25	6	17	3	14	25	6	17	3	14	25
2	13	24	10	16	2	13	24	10	16	2
9	20	1	12	23	9	20	1	12	23	9
11	22	8	19	5	11	22	8	19	5	11

Fig. 11.

Every square of every 25 of these numbers, as for example the two dark-bordered ones, possesses the property that the addition of the horizontal, vertical, and diagonal rows gives each the same sum, 65.

As an example of a higher number of cells we will append here a magic square of 11 times 11 spaces formed by the general method of De la Hire from the two auxiliary squares of Figs. 12 and 13 From these two auxiliary squares we obtain by the addition of the two numbers of every two similarly situated cells, the magic square, exhibited in Diagram 14, in which each row gives the same sum 671.

1	2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	1	2
5	6	7	8	9	10	11	1	2	3	4
7	8	9	10	11	1	2	3	4	5	6
9	10	11	1	2	3	4	5	6	7	8
11	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11	1
4	5	6	7	8	9	10	11	1	2	3
6	7	8	9	10	11	1	2	3	4	5
8	9	10	11	1	2	3	4	5	6	7
10	11	1	2	3	4	5	6	7	8	9

Fig. 12.

0	11	22	33	44	55	66	77	88	99	110
33	44	55	66	77	88	99	110	0	11	22
66	77	88	99	110	0	11	22	33	44	55
99	110	0	11	22	33	44	55	66	77	88
11	22	33	44	55	66	77	88	99	110	0
44	55	66	77	88	99	110	0	11	22	33
77	88	99	110	0	11	22	33	44	55	66
110	0	11	22	33	44	55	66	77	88	99
22	33	44	55	66	77	88	99	110	0	11
55	66	77	88	99	110	0	11	22	33	44
88	99	110	0	11	22	33	44	55	66	77

Fig. 13.

1	13	25	37	49	61	73	85	97	109	121
36	48	60	72	84	96	108	120	11	12	24
71	83	95	107	119	10	22	23	35	47	59
106	118	9	21	33	34	46	58	70	82	94
20	32	44	45	57	69	81	93	105	117	8
55	56	68	80	92	104	116	7	19	31	43
79	91	103	115	6	18	30	42	54	66	67
114	5	17	29	41	53	65	77	78	90	102
28	40	52	64	76	88	89	101	113	4	16
63	75	87	99	100	112	3	15	27	39	51
98	110	111	2	14	26	38	50	62	74	86

Fig. 14.

IV.

EVEN-NUMBERED SQUARES.

Of magic squares having an even number of places we have hitherto had to deal only with the square of 4. To construct squares of this description having a higher even number of places, different and more complicated methods must be employed than for squares of odd numbers of places. However, in this case also, as in dealing with the square of 4, we start with the natural sequence of the numbers and must then find the complements of the numbers with respect to some other certain number (as 17 in the square of 4) and also effect certain exchanges of the numbers with one another. To form, for example, a magic square of 6 times 6 places, we inscribe in the 12 diagonal cells the numbers that in the natural sequence of inscription fall into these places, then in the remaining cells the complements of the numbers that belong therein with respect to 37, and finally effect the following six exchanges, viz. of the numbers 33 and 3, 25 and 7, 20 and 14, 18 and 13, 10 and 9, and 5 and 2. In this way the following magic square is obtained.

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Fig. 15.

This square may also be constructed by the method of De la Hire, from two auxiliary squares with the numbers 1, 2, 3, 4, 5, 6 and 0, 6, 12, 18, 24, 30 respectively. In this case, however, the vertical rows of the one square and the horizontal rows of the other must each so contain two same numbers thrice repeated that the summation shall always remain 21 and 90 respectively. In this manner we get the magic square last given above from the two following auxiliary squares:

1	5	4	3	2	6
6	2	4	3	5	1
6	5	3	4	2	1
1	5	3	4	2	6
6	2	3	4	5	1
1	2	4	3	5	6

Fig. 16.

and

0	30	30	0	30	0
24	6	24	24	6	6
18	18	12	12	12	18
12	12	18	18	18	12
6	24	6	6	24	24
30	0	0	30	0	30

Fig. 17.

It is to be noted in connection with this example that here also as in the case of odd-numbered squares, it is possible so to inscribe six times the numbers from 1 to 6 that each number shall appear once and only once in each horizontal, vertical, and diagonal row; for example, in the following manner:

1	2	3	4	5	6
2	4	6	1	3	5
3	6	5	2	1	4
5	3	1	6	4	2
6	5	4	3	2	1
4	1	2	5	6	3

Fig. 18.

But if we attempt so to insert, in a like manner, the other set of numbers 0, 6, 12, 18, 24, 30 in a second auxiliary square, that each number of the first auxiliary square shall stand once and once only in a corresponding cell with each number of the second square, all the attempts we may make to fulfil coincidently the last named condition will result in failure. It is therefore necessary to select auxiliary squares like the two given above. It is noteworthy, that the fulfilment of the second condition is impossible only in the case of the square of 6, but that in the case of the square of 4 or of the square of 8, for example, two auxiliary squares, such as the method of De la Hire requires, are possible. Thus, taking the square of 4 we get

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

Fig. 19.

and

0	4	8	12
8	12	0	4
12	8	4	0
4	0	12	8

Fig. 20.

The reader may form for himself the magic square which these give.

The existence of these two auxiliary squares furnishes a key to the solution of a pretty problem at cards. If we replace, namely, the numbers 1, 2, 3, 4 by the Ace, the King, the Queen, and the Knave, and the numbers 0, 4, 8, 12 by the four suits, clubs, spades, hearts, and diamonds, we shall at once perceive that it is possible, and must be so necessarily, quadratically to arrange in such a manner the four Aces, the four Kings, the Four Queens, and the four

Knaves, that in each horizontal, vertical, and diagonal row, each one of the four suits and each one of the four denominations shall appear once and once only. The auxiliary squares above given furnish the appended solution of this problem :

CLUBS ACE	SPADES KING	HEARTS QUEEN	DIAMONDS KNAVE
HEARTS KNAVE	DIAMONDS QUEEN	CLUBS KING	SPADES ACE
DIAMONDS KING	HEARTS ACE	SPADES KNAVE	CLUBS QUEEN
SPADES QUEEN	CLUBS KNAVE	DIAMONDS ACE	HEARTS KING

Fig. 21.

To fix the solution of the problem in the memory, observe that, starting from the several corners, each suit and each denomination must be placed in the spots of the move of a Knight. If we fix the positions of the four cards of any one row, there will be only two possibilities left of so placing the other cards that the required condition of having each suit and each denomination once and only once in each row shall be fulfilled.

Of magic squares of an even number of places we have up to this point examined only the squares of 4 and of 6. For the sake of completeness we append here one of 8 and one of 10 places. The mode of construction of these squares is similar to the method above discussed for the lower even numbers.

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Fig. 22.

1	99	3	97	96	5	94	8	92	10
90	12	88	14	86	85	17	83	19	11
80	79	23	77	25	26	74	28	22	71
31	69	68	34	66	65	37	33	62	40
60	42	58	57	45	46	44	53	49	51
50	52	43	47	55	56	54	48	59	41
61	32	38	64	36	35	67	63	39	70
21	29	73	27	75	76	24	78	72	30
20	82	18	84	15	16	87	13	89	81
91	9	93	4	6	95	7	98	2	100

Fig. 23.

The magic squares of even numbers thus constructed are not the only possible ones. On the contrary, there are very many others possible, which obey different laws of formation. It has been calculated, for example, that with the square of 4 it is possible to construct 880, and with the square of 6, *several million*, different magic squares. The number of odd-numbered magic squares constructible by the method of De la Hire is also very great. With the square of 7, the possible constructions amount to 363,916,800. With the squares of higher numbers the multitude of the possibilities increases in the same enormous ratio.

V.

MAGIC SQUARES WHOSE SUMMATION GIVES THE NUMBER OF A YEAR.

The magic squares which we have so far considered contain only the natural numbers from 1 upwards. It is possible, however, easily to deduce from a correct magic square other squares in which a different law controls the sequence of the numbers to be inscribed. Of the squares obtained in this manner, we shall devote our attention here only to such in which, although formed by the inscription of successive numbers, the sum obtained from the addition of the rows is a determinate number which we have fixed upon beforehand, as *the number of a year*. In such a case we have simply to add to the numbers of the original square a determinate number so to be calculated, that the required sum shall each time appear. If this sum is divisible by 3, magic squares will always be obtainable with 3 times 3 spaces which shall give this sum. In such a case we divide the sum required by 3 and subtract 5 from the result in order to obtain the number which we have to add to each number of the original square. If the sum desired is even but not divisible by 4, we must then subtract from it 34 and take one fourth of the result, to obtain the number which in this case is to be added in each place. If, for example, we wish to obtain the number of the year 1890 as the resulting sum of each row, we shall have to add to each of the numbers of an ordinary magic square of 4 times 4 spaces the number 464; in other words, instead of the numbers from 1 to 16 we have to insert in the squares the numbers from 465 to 480. As the number of the present year 1892 is divisible by 11, it must be possible to deduce from the magic square constructed by us at the conclusion of Section III a second magic square in which each row of 11 cells will give the number of the year 1892. To do this, we subtract from 1892 the sum of the original square, namely 671, and divide the remainder by 11, whereby we get 111 and thus perceive that the numbers from 112 to 232 are to be inscribed in the cells of

112	124	136	148	160	172	184	196	208	220	232	=1892
147	159	171	183	195	207	219	231	122	123	135	=1892
182	194	206	218	230	121	133	134	146	158	170	=1892
217	229	120	132	144	145	157	169	181	193	205	=1892
131	143	155	156	168	180	192	204	216	228	119	=1892
166	167	179	191	203	215	227	118	130	142	154	=1892
190	202	214	226	117	129	141	153	165	177	178	=1892
225	116	128	140	152	164	176	188	189	201	213	=1892
139	151	163	175	187	199	200	212	224	115	127	=1892
174	186	198	210	211	223	114	126	138	150	162	=1892
209	221	222	113	125	137	149	161	173	185	197	=1892
1892	1892	1892	1892	1892	1892	1892	1892	1892	1892	1892	

Fig. 24.

the square required. We get in this way the preceding square, from which *one and the same sum, namely 1892, can be obtained 44 times,* first from each of the 11 horizontal rows, secondly from each of the 11 vertical rows, thirdly from each of the two diagonal rows, and fourthly twenty additional times from each and every pair of any two rows that lie parallel to a diagonal, have together 11 cells, and lie on different sides of the diagonal, as for example, 196, 122, 158, 205, 131, 167, 214, 140, 187, 223, 149.

VI.

CONCENTRIC MAGIC SQUARES.

The acuteness of the mathematicians has also discovered magic squares which possess the peculiar property that if one row after another be taken away from each side, the smaller inner squares remaining will still be magical squares, that is to say, all their rows when added will give the same sum. It will be sufficient to give two examples here of such squares, (the laws for their construction being somewhat more complicated,) of which the first has 7 times 7 and the second 8 times 8 places. The numbers within each of the dark-bordered frames form with respect to the centre smaller squares which in their own turn are magical.

4	5	6	43	39	38	40
49	15	16	33	30	31	1
48	37	22	27	26	13	2
47	36	29	25	21	14	3
8	18	24	23	28	32	42
9	19	34	17	20	35	41
10	45	44	7	11	12	46

Fig. 25.

1	56	55	11	53	13	14	57
63	15	47	22	42	24	45	2
62	49	25	40	34	31	16	3
4	48	28	37	35	30	17	61
5	44	39	26	32	33	21	60
59	19	38	27	29	36	46	6
58	20	18	43	23	41	50	7
8	9	10	54	12	52	51	64

Fig. 26.

In the first of these two squares the internal square of 3 times 3 places contains the numbers from 21 to 29 in such a manner that each row gives when added the sum of 75. This square lies within a larger one of 5 times 5 spaces, which contains the numbers from 13 to 37 in such a manner that each row gives the sum of 125. Finally, this last square forms part of a square of 7 times 7 places which contains the numbers from 1 to 49 so that each row gives the sum of 175.

In the second square the inner central square of 4 times 4 places contains the numbers from 25 to 40 in such a manner that each row gives the sum of 130. This square is the middle of a square of 6 times 6 places which so contains the numbers from 15 to 50 that each row gives the sum 165. Finally, this last square is again the middle of an ordinary magic square composed of the numbers from 1 to 64.

VII.

MAGICAL SQUARES WITH MAGICAL PARTS.

If we divide a square of 8 times 8 places by means of the two middle lines parallel to its sides into 4 parts containing each 4 times 4 spaces, we may propound the problem of so inserting the numbers from 1 to 64 in these spaces that not only the whole shall form a magic square, but also that each of the 4 parts individually shall be magical, that is to say, give the same sum for each row. This problem also has been successfully solved, as the following diagram will show.

1	4	63	62	5	8	59	58
64	61	2	3	60	57	6	7
42	43	24	21	34	35	32	29
23	22	41	44	31	30	33	36
13	16	51	50	9	12	55	54
52	49	14	15	56	53	10	11
38	39	28	25	46	47	20	17
27	26	37	40	19	18	45	48

Fig. 27.

The 4 numbers in each row of any one of the sub-squares here, gives 130; so that the sum of each one of the rows of the large square will be 260.

Finally, in further illustration of this idea, we will submit to the consideration of our readers a very remarkable square of the numbers from 1 to 81. This square, which will be found on the following page (Fig. 28), is divided by parallel lines into 9 parts, of which each contains 9 consecutive numbers that severally make up a magic square by themselves.

Wonderful as the properties of this square may appear, the law by which the author constructed it is equally simple. We have simply to regard the 9 parts as the 9 cells of a magic square of the numbers from I to IX, and then to inscribe by the magic prescript in the square designated as I the numbers from 1 to 9, in the square designated as II the numbers from 10 to 18, and so on. In this way the square above given is obtained from the following base-square :

31	36	29	76	81	74	13	18	11
30	32	34	75	77	79	12	14	16
35	28	33	80	73	78	17	10	15
22	27	20	40	45	38	58	63	56
21	23	25	39	41	43	57	59	61
26	19	24	44	37	42	62	55	60
67	72	65	4	9	2	49	54	47
66	68	70	3	5	7	48	50	52
71	64	69	8	1	6	53	46	51

Fig. 28.

IV	IX	II
III	V	VII
VIII	I	VI

Fig. 29.

VIII.

MAGIC SQUARES THAT INVOLVE THE MOVE OF THE CHESS-KNIGHT.

What one of our readers does not know the problems contained in the recreation columns of our magazines, the requirements of which are to compose into a verse 8 times 8 quadratically arranged syllables, of which every two successive syllables stand on spots so situated with respect to each other that a chess-knight can move from the one to the other? If we replace in such an arrangement the 64 successive syllables by the 64 numbers from 1 to 64, we shall obtain a knight-problem made up of numbers. Methods also exist indeed for the construction of such dispositions of numbers, which then form the foundation of the construction of the problems in the newspapers. But the majority of knight-problems of this class are the outcome of experiment rather than the product of methodical creation. If however it is a severe test of patience to form a knight-problem by experiment, it stands to reason that it is a still severer trial to effect at the same time the additional result that the 64 numbers which form the knight-problem shall also form a magic square.

This trial of endurance was undertaken several decades ago, by a pensioned Moravian officer named Wenzelides, who was spending the last days of his life in the country. After a series of trials which lasted years he finally succeeded in so inscribing in the 64 squares of the chess-board the numbers from 1 to 64 that successive numbers, as well also as the numbers 64 and 1, were always removed from one another in distance and direction by the move of a knight, and that in addition thereto the summation of the horizontal and the vertical rows always gave the same sum 260. Ultimately he discovered several squares of this description, which were published in the *Berlin Chess Journal*. One of these is here appended :

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

Fig. 30.

The move of the knight and the equality of the summation of the horizontal and vertical rows, therefore, are the facts to be noted here. The diagonal rows do *not* give the sum 260. Perhaps some one among our readers who possesses the time and patience will be tempted to outdo Wenzelides, and to devise a numeral knight-problem of this kind which will give 260 not only in the horizontal and vertical but also in the two diagonal rows.

IX.

MAGICAL POLYGONS.

So far we have only considered such extensions of the idea underlying the construction of the magic square in which the figure of the square was retained. We may however contrive extensions of the idea in which instead of a square, a rectangle, a triangle, or a pentagon, and the like, appear. Without entering into the consideration of the methods for the construction of such figures, we will give here of magical polygons simply a few examples, all supplied by Professor Scheffler:

1) The numbers from 1 to 32 admit of being written in a rectangle of 4 × 8 in such a manner that the long horizontal rows give the sum of 132 and the short vertical rows the sum of 66; thus:

1	10	11	29	28	19	18	16
9	2	30	12	20	27	7	25
24	31	3	21	13	6	26	8
32	23	22	4	5	14	15	17

Fig. 31.

2) The numbers from 1 to 27 admit of being so arranged in three regular triangles about a point which forms a common centre, that each side of the outermost triangle will present 6 numbers of the total summation 96 and each side of the middle triangle 4 numbers whose sum is 61; as the following figure shows:

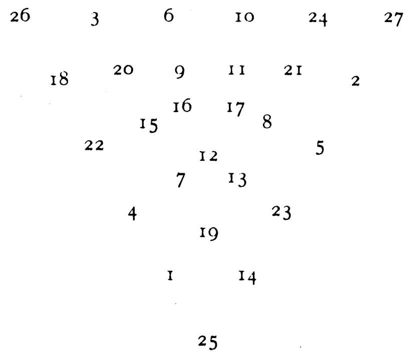


Fig. 32.

3) The numbers from 1 to 80 admit of being formed about a point as common centre into 4 pentagons, such that each side of the first pentagon from within contains two numbers, each side of the second pentagon four numbers, each of the third six numbers, and each side of the fourth, outermost pentagon eight numbers. The sum of the numbers of each side of the second pentagon is 122, the sum of those of each side of the third pentagon is 248, and that of those of each side of the fourth pentagon 254. Furthermore, the sum of any four corner numbers lying in the same straight line with the centre, is also the same; namely, 92.

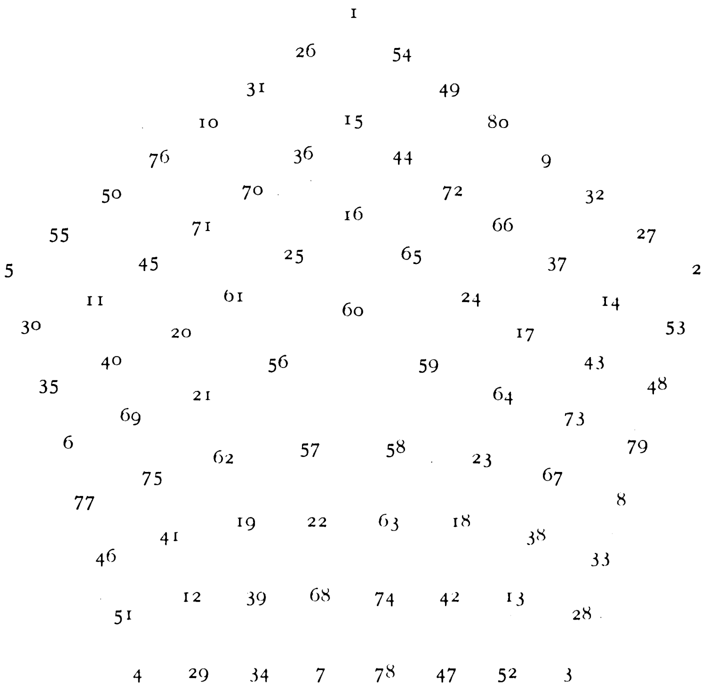


Fig. 33.

4) The numbers from 1 to 73 admit of being arranged about a centre, in which the number 37 is written, into three hexagons which contain respectively 3, 5, and 7 numbers in each side and possess the following pretty properties. Each hexagon always gives the same sum, not only when the summation is made along its six sides, but also when it is made along the six diameters that join its corners

and along the six that are constructed at right angles to its sides; this sum, for the first hexagon from within, is 111, for the second 185, and for the third 259.

```
            1    5   6   70  60  59  58
          63                           8
        62       19  53  46  22  45      9
      61       20                  24      64
    2        48      31  42  38      49      57
  3        47      39          40      44      56
 67     51       41      37      33      23      7
  66       50      34          35      54      11
    65       25      36  32  43      26      12
       10      30                  27      13
         17      29  21  28  52  55      72
           18                          71
             16  69  68  4   14  15  73
```

Fig. 34.

X.

MAGIC CUBES.

Several inquirers, particularly Kochansky (1686), Sauveur (1710), Hugel (1859), and Scheffler (1882), have extended the principle of the magic squares of the plane to three-dimensioned space. Imagine a cube divided by planes parallel to its sides and equidistant from one another, into cubical compartments. The problem is then, so to insert in these compartments the successive natural numbers that every row from the right to the left, every row from the front to the back, every row from the top to the bottom, every diagonal of a square, and every principal diagonal passing through the centre of the cube shall contain numbers whose sum is always the same. For 3 times 3 times 3 compartments, a magic cube of this description is not constructible. For 4 times, 4 times 4 compartments a cube is constructible such that any row parallel to an edge of the cube and every principal diagonal give the sum of 130. To obtain a magic cube of 64 compartments, imagine the numbers which belong in the compartments written on the upper surface of the same and the numbers then taken off in layers of 16 from the top downwards. We obtain thus 4 squares of 16 cells each, which together make up the magic cube; as the following diagrams will show:

First Layer
from the Top.

1	48	32	49
60	21	37	12
56	25	41	8
13	36	20	61

Second Layer
from the Top.

63	18	34	15
6	43	27	54
10	39	23	58
51	30	46	3

Third Layer
from the Top.

62	19	35	14
7	42	26	55
11	38	22	59
50	31	47	2

Fourth Layer
from the Top.

4	45	29	52
57	24	40	9
53	28	44	5
16	33	17	64

The same sum 130 here comes out not less than 52 times; viz. in the first place from the 16 rows from left to right, secondly from the 16 rows from the front to the back, thirdly from the 16 rows counting from the top to the bottom, and lastly from the 4 rows which join each two opposite corners of the cube, namely from the rows: 1, 43, 22, 64; 49, 27, 38, 16; 13, 39, 26, 52; 61, 23, 42, 4.

For a cube with 5 compartments in each edge the arrangement of the figures can so be made that all the 75 rows parallel to any and every edge, all the 30 rows lying in any diagonal of a square, and all the 4 rows forming any principal diagonal shall have one and the same summation, 315.

Just as the magic squares of an odd number of cells could be formed with the aid of *two* auxiliary squares, so also odd-numbered magic cubes can be constructed with the help of *three* auxiliary cubes.

First Layer from Top.

121	27	83	14	70
10	61	117	48	79
44	100	1	57	113
53	109	40	91	22
87	18	74	105	31

Second Layer from Top.

2	58	114	45	96
36	92	23	54	110
75	101	32	88	19
84	15	66	122	28
118	49	80	6	62

Third Layer from Top.

33	89	20	71	102
67	123	29	85	11
76	7	63	119	50
115	41	97	3	59
24	55	106	37	93

Fourth Layer from Top.

64	120	46	77	8
98	4	60	111	42
107	38	94	25	51
16	72	103	34	90
30	81	12	68	124

Lowest Layer.

95	21	52	108	39
104	35	86	17	73
13	69	125	26	82
47	78	9	65	116
56	112	43	99	5

In this manner the preceding magic cube of 5 times 5 times 5 compartments is formed, in which, it may be additionally noticed, the middle number between 1 and 125, namely 63, is placed in the central compartment; by which arrangement the attainment of the sum of 315 is assured in the four principal diagonals and the 30 sub-diagonals. The condition attained in the magic squares, that the diagonal-pairs parallel to the sub-diagonals also shall give the sum 315 is not attainable in this case but is so in the case of higher numbers of compartments.

CONCLUSION.

Musing on such problems as are the magic squares is fascinating to thinkers of a mathematical turn of mind. We take delight in discovering a harmony that abides as an intrinsic quality in the forms of our thought. The problems of the magic squares are playful puzzles, invented as it seems for mere pastime and sport. But there is a deeper problem underlying all these little riddles, and this deeper problem is of a sweeping significance. It is the philosophical problem of the world-order.

The formal sciences are creations of the mind. We build the sciences of mathematics, geometry, and algebra with our conception of pure forms which are abstract ideas. And the same order that prevails in these mental constructions permeates the universe, so that an old philosopher, overwhelmed with the grandeur of law, imagined he heard its rhythm in a cosmic harmony of the spheres.

H. Schubert.